On the MIMO Capacity with Joint Sum and Per-antenna Power Constraints: A New Efficient Numerical Method

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***Abstract*—We propose a semi-closed-form solution to the problem of computing the capacity and optimal signaling for a Gaussian MIMO channel under a joint sum power constraint (SPC) and per-antenna power constraint (PAPC). Existing efficient solutions to this fundamental problem are only applicable to some special cases: multiple-input single-output (MISO) systems, or full column rank MIMO channels with sufficiently high transmit power, or full-rank optimal signaling. For the general case, we present an efficient numerical method to solve the considered problem which does not make any assumptions on the rank of the channel matrix or the maximum transmit power. To achieve this, the considered problem is transformed into an equivalent minimax problem. We then exploit the special structure of the minimax problem to derive a closed-form solution based on a concave-convex procedure (CCP)-like algorithm. Extensive simulation results show that our proposed algorithm outperforms the existing solutions in terms of complexity and generality.**

***Index Terms*—MIMO, sum power constraint, per-antenna power constraint, minimax, concave-convex procedure.**

I. Introduction

The capacity of a Gaussian multiple-input multiple-output (MIMO) channel under a sum power constraint (SPC) or per-antenna power constraint (PAPC) has been studied extensively [1]–[7]. While the former can be due to power budget or regulations, the latter, which is more realistic, is considered to avoid/mitigate nonlinear distortion caused by the power amplifier associated with each transmit antenna. Efficient solutions to the computation of MIMO capacity with either SPC or PAPC have been well-studied [1]–[3], [6]–[8].

In practice, other types of power constraint can also be imposed on a MIMO system, not necessarily limited to an individual SPC and/or PAPC. For example, in the context of cognitive radio networks, interference temperature constraints can be imposed on a secondary user (SU) to limit the interference generated at a primary user (PU) [9]–[11]. All of these constraints, including SPC and PAPC, can be generally modeled as linear transmit covariance constraints (cf. [9], [12], [13] for further details), for which several numerical methods were proposed [9], [12]–[14].

A. Related Work

In this paper, we consider the capacity of Gaussian MIMO systems for the specific case of joint SPC and PAPC, which was previously studied in [13]–[18]. Here we provide an overview of these related papers. The work of [15] is only applicable to multiple-input single-output (MISO) systems, while that of [18] only *partially* addresses the general MIMO case. For the case of MIMO, Cao *et al.* presented in [16] an iterative solution, solving a sequence of MIMO capacity problems with PAPC. The main idea of this method is to make use of the closed-form solution for MIMO capacity with PAPC proposed in [19]. However, the closed-form solution in [19] is suboptimal for the MIMO capacity problem with PAPC, unless the channel has full column rank and the system operates in the high signal-to-noise ratio (SNR) regime. In [17], Loyka proposed a closed-form solution, which is, unfortunately, only applicable to full column rank channels, operation in the high-SNR regime, and an equal power constraint on all transmit antennas. Under these conditions, a closed-form solution for the optimal covariance matrix is possible by solving the KKT conditions of their considered problem [17]. In [13], we presented an iterative method based on the Gauss-Seidel method to solve the considered problem. More recently, in [14], Chaluvadi *et al.* proposed a closed-form solution for MIMO channels with per-group power constraints which include PAPC as a special case by treating each antenna as a separate group. However, this closed-form solution is only applicable when the channel is of full column rank and the optimal signaling also needs to be full rank. For the case when the MIMO channel is not of full column rank, a projected factored gradient descent algorithm (PFGD) was also presented in [14]. The main drawback of the PFGD algorithm is that it only guarantees local convergence under some conditions on the SPC and PAPC since it is based on a non-convex formulation, which is also verified by our numerical experiments to be presented in Section IV.

B. Contributions

It is obvious from the above discussions that a closed-form solution for MIMO capacity with joint SPC and PAPC is only possible for some special cases. In this paper, we propose an efficient semi-closed-form approach to computing the capacity and optimal signaling of a Gaussian MIMO channel under joint SPC and PAPC, where no assumptions

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are made regarding the rank of the underlying channel or on the SPC and PAPC. To the best of the authors' knowledge, no closed-form solution for this general case has been reported in previous publications including [14], [16], [17], and thus an efficient numerical method is still desired. To this end, our contributions are summarized as follows:

- Different from existing methods, we first invoke an equivalent minimax problem of the MIMO capacity problem with joint SPC and PAPC and exploit the special structure of the resulting minimax problem.
- We then combine AO and CCP to derive a semi-closed-form solution to the general case where no assumptions on the rank of the underlying channel or on SPC and PAPC are made. In particular, we provide a closed-form solution for the minimization step by considering an upper bound of the objective.
- We provide numerical results on the capacity of MIMO systems in comparison with existing methods under joint SPC and PAPC which have not been reported previously.

Compared to the numerical methods presented in our previous works of [12], [13] which are also derived based on the equivalent minimax formulation, the differences between them and the one proposed in this paper are clarified as follows. In [12], we simply treat SPC and PAPC as general linear transmit covariance constraints and thus the special underlying structure of the considered problem is not exploited. Then a gradient-projection-based method is adopted to solve the minimization step. In [13], we presented a nonlinear Gauss–Seidel method in combination with the method of Lagrange multipliers to solve the minimization step. More precisely, when the Lagrange multiplier is fixed, the Gauss-Seidel iteration is used to compute the Lagrangian dual function, and then a bisection or Newton step is carried out to find the optimal Lagrange multiplier. Thus, the method presented in [13] is essentially a double-loop iterative method. In contrast, in this paper, we tackle the MIMO capacity problem with joint SPC and PAPC from the view of solving the KKT conditions directly, which results in a method based on *semi-closed-form expressions*. Further details on the differences are provided in Section III.

Notation: Standard notations are used in this paper. Bold lower and upper case letters represent vectors and matrices, respectively. $\mathbf{I}$ and $\mathbf{0}$ define identity and zero matrices respectively, of which the size can be easily inferred from the context. $\mathbb{C}^{M\times N}$ denotes the space of $M\times N$ complex matrices; $\mathbf{H}^\dagger$ and $\mathbf{H}^T$ denote the Hermitian and ordinary transpose of $\mathbf{H}$, respectively; $\operatorname{diag}(\mathbf{x})$ denotes the diagonal matrix having diagonal entries matching the vector $\mathbf{x}$; $\operatorname{tr}(\mathbf{H})$ denotes the trace of $\mathbf{H}$. Furthermore, we denote the expected value of a random variable by $\mathbb{E}[.]$, the determinant by $|\cdot|$, and $[x]_+ = \max(x,0)$.

II. System Model

We consider a Gaussian MIMO channel, where the transmitter and the receiver are equipped with N and M antennas, respectively. The channel state information is assumed to be perfectly known at the transmitter and the receiver. The received signal is given by

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{z} \tag{1}$$

where $\mathbf{H}\in\mathbb{C}^{M\times N}$ is the channel matrix, $\mathbf{x}\in\mathbb{C}^{N\times 1}$ is the vector of transmitted symbols, and $\mathbf{z}\in\mathbb{C}^{M\times 1}$ is the additive noise with distribution $\mathcal{CN}(\mathbf{0},\mathbf{I}_M)$. Let $\mathbf{X}=\mathbb{E}[\mathbf{x}\mathbf{x}^\dagger]$ be the input covariance matrix for the transmitted signal. We are interested in finding the capacity and optimal signaling of the above channel with joint SPC and PAPC, which is formulated as[1]

$$\begin{aligned}
&\underset{\mathbf{X}\succeq\mathbf{0}}{\text{maximize}} && \log|\mathbf{I}+\mathbf{H}\mathbf{X}\mathbf{H}^\dagger| && \text{(2a)}\\
&\text{subject to} && \operatorname{tr}(\mathbf{X})\le P_0 && \text{(2b)}\\
& && [\mathbf{X}]_{i,i}\le P_i,\ i=1,2,\dots,N && \text{(2c)}
\end{aligned}$$

where P_0 is the maximum total power budget, and P_i is the maximum allowable power at the ith transmit antenna. Thus, the constraints (2b) and (2c) are called the SPC and PAPC, respectively.

Remark 1. We can reformulate (2) as a standard semidefinite program (SDP) and then use an off-the-shelf SDP optimization software to find the optimal transmit covariance matrix. However, the complexity of such a method, which is usually based on interior-point algorithms, increases dramatically with the problem size, and thus is not suitable for large-scale MIMO systems.

III. Proposed CCP-like Algorithm

In this section, we present an efficient numerical method to find the capacity and optimal signaling for a MIMO channel subject to joint SPC and PAPC. We remark that previous research has aimed to solve this problem by working directly on (2) [14]–[18]. In contrast, our proposed solution is derived based on a minimax reformulation of (2).

A. Proposed Approach

Let us denote by λ_0 the Lagrange multiplier for the SPC (2b) and $\hat{\boldsymbol{\lambda}} = [\lambda_1,\lambda_2,\dots,\lambda_N]^T$ the vector of the Lagrange multipliers for the constraints (2c). Furthermore, let $\mathbf{p}=[P_0,P_1,P_2,\dots,P_N]^T$ be the vector stacking the power thresholds corresponding to the SPC and PAPC. Following the duality transformation in [4], [7], [13], (2) can be transformed into the following equivalent minimax problem:

$$\begin{aligned}
&\min_{\boldsymbol{\Lambda}\succ\mathbf{0}}\ \max_{\bar{\mathbf{X}}\succeq\mathbf{0}} && \log\frac{|\boldsymbol{\Lambda}+\mathbf{H}^\dagger\bar{\mathbf{X}}\mathbf{H}|}{|\boldsymbol{\Lambda}|}\triangleq f(\boldsymbol{\Lambda},\bar{\mathbf{X}})\\
&\text{subject to} && \operatorname{tr}(\bar{\mathbf{X}})=P,\mathbf{p}^T\boldsymbol{\lambda}=P
\end{aligned}\tag{3}$$

where $P=\sum_{i=0}^N P_i$, $\boldsymbol{\lambda}=[\lambda_0,\hat{\boldsymbol{\lambda}}^T]^T\ge 0$, and $\boldsymbol{\Lambda}$ is a *diagonal* matrix defined as

$$\boldsymbol{\Lambda}\triangleq\operatorname{diag}\big(\lambda_0+\lambda_1,\lambda_0+\lambda_2,\dots,\lambda_0+\lambda_N\big). \tag{4}$$

To describe the proposed iterative algorithm, we denote by $\boldsymbol{\Lambda}_n$ the value of $\boldsymbol{\Lambda}$ achieved at iteration n. The proposed CCP-like algorithm alternately optimizes $\boldsymbol{\Lambda}$ and $\bar{\mathbf{X}}$ as follows:

- For a given $\boldsymbol{\Lambda}_n$, we solve the maximization with respect to $\bar{\mathbf{X}}$ to find $\bar{\mathbf{X}}_n$:

$$\bar{\mathbf{X}}_n = \underset{\bar{\mathbf{X}}\succeq\mathbf{0};\operatorname{tr}(\bar{\mathbf{X}})=P}{\arg\max}\ \log\big|\mathbf{I}+\boldsymbol{\Lambda}_n^{-1/2}\mathbf{H}^\dagger\bar{\mathbf{X}}\mathbf{H}\boldsymbol{\Lambda}_n^{-1/2}\big|. \tag{5}$$

[1] Note that we normalize the actual channel matrix with respect to the noise power, and thus the noise power is absorbed in $\mathbf{H}$.

The above problem admits a water-filling solution [1], [7], [8]. We skip the details for the sake of brevity.

- For a given $\bar{\mathbf{X}}_n$, we consider the minimization over $\mathbf{\Lambda}$ to find $\mathbf{\Lambda}_{n+1}$. If exact minimization for the $\mathbf{\Lambda}$ update is used, the ping-pong effect can occur. The reason is that the maximization over $\bar{\mathbf{X}}$ can increase the objective to an extreme point (can be thought of as a maximum), while the minimization over $\mathbf{\Lambda}$ will decrease the objective to another extreme point (can be thought of as a minimum). Thus, after some iterations, this alternating optimization process may fluctuate between two extreme points, and thus never converges (cf. [4], [7, Fig. 2]). To avoid the ping-pong effect, we consider the upper bound of $f(\mathbf{\Lambda}, \bar{\mathbf{X}})$ obtained by applying CCP to the $\log|\cdot|$ function in $f(\mathbf{\Lambda}, \bar{\mathbf{X}})$ [13], [20] for the $\mathbf{\Lambda}$-minimization, and thus $\mathbf{\Lambda}_{n+1}$ is found to be the optimal solution to the following problem

$$\begin{array}{ll} \underset{\mathbf{\Lambda}\succ\mathbf{0}}{\text{minimize}} & \text{tr}\big(\mathbf{\Phi}_n^{-1}\mathbf{\Lambda}\big) - \log|\mathbf{\Lambda}| \\ \text{subject to} & \mathbf{p}^T\boldsymbol{\lambda} = P \end{array} \tag{6}$$

where $\mathbf{\Phi}_n = \mathbf{\Lambda}_n + \mathbf{H}^\dagger\bar{\mathbf{X}}_n\mathbf{H}$. It is worth noting that a CCP-based method generates a local optimum in general. Since our original objective is convex with respect to $\mathbf{\Lambda}$, thus the local minimum is also the global minimum [7]. Recall that $\mathbf{\Lambda}$ is diagonal and is defined in (4), and thus (6) is reduced to

$$\begin{array}{ll} \underset{\boldsymbol{\lambda}\geq 0}{\text{minimize}} & \sum_{i=1}^{N}\big(\lambda_0+\lambda_i\big)\phi_{n,i} - \sum_{i=1}^{N}\log(\lambda_0+\lambda_i) \\ \text{subject to} & \sum_{i=0}^{N}\lambda_i P_i = P \end{array} \tag{7}$$

where $\phi_{n,i} = \big[\mathbf{\Phi}_n^{-1}\big]_{i,i}$.

The idea of using the upper bound when minimizing over $\mathbf{\Lambda}$ is to make sure that the maximization over $\bar{\mathbf{X}}$, which will be performed in the next step, cannot bring the objective back to the previous value. In this way, the monotonic decrease of the objective sequence is always achieved, leading to the convergence of the proposed algorithm. The following two remarks are in order.

Remark 2. Before proceeding further we now elaborate why considering joint SPC and PAPC is much more challenging, compared to the PAPC only, from the perspective of finding an analytical solution to (7). Indeed, problem (7) looks very similar to [7, Eqn. (26)] and thus a question arising naturally is if the method presented in [7], which is dedicated to the PAPC only, is applicable to solving (7). To appreciate the challenges, let us now consider the partial Lagrangian function of (7) which is given by

$$\begin{aligned} \tilde{L}(\boldsymbol{\lambda},\gamma) = &\sum\nolimits_{i=1}^{N}\big(\lambda_0+\lambda_i\big)\phi_{n,i} - \sum\nolimits_{i=1}^{N}\log(\lambda_0+\lambda_i) \\ &+ \gamma\big(\sum\nolimits_{i=0}^{N}\lambda_i P_i - P\big). \end{aligned} \tag{8}$$

It is easy to see that if the SPC is not present, we can omit λ_0 in (8) and $\tilde{L}(\boldsymbol{\lambda},\gamma)$ becomes *separable* in terms of λ_i. Thus, the solution to computing the dual function $g(\gamma) = \min_{\boldsymbol{\lambda}} \tilde{L}(\boldsymbol{\lambda},\gamma)$ is $\lambda_i = \frac{1}{\phi_{n,i}+\gamma P_i}$ and the optimal γ for the dual problem can be found by a bisection search or Newton's method. This is indeed the algorithm presented in [7].

However, for the joint SPC and PAPC, a closed-form expression for each λ_i cannot be achieved for solving the dual problem due to the coupling of λ_0 and other λ_i's, $i \geq 1$. To deal with this, we presented in [13] a double-loop iterative method to solve the dual problem. More specifically, for a given γ, we apply the Gauss-Seidel iteration where each λ_i is sequentially optimized while the others are fixed. Note that this step admits a closed-form solution (cf. [13, (15)-(16)]). After the Gauss-Seidel iteration converges, we then use a bisection search or Newton's method to solve the dual problem, similarly as done in [7].

Remark 3. In this paper, we only consider the non-trivial case where $\min\{P_i\} < P_0 < \hat{P} = \sum_{i=1}^{N} P_i$. As a result, the SPC must be binding as proved in Proposition 1 below. In this regard the proposed method in this paper will be only applicable to this non-trivial case. We note that if $P_0 \leq \min_{1\leq i\leq N}\{P_i\}$, it is easy to see that the PAPC is inactive for all antennas and (2) reduces to the MIMO capacity with a single SPC. Similarly, if $P_0 \geq \hat{P}$, the SPC can be omitted and thus (2) becomes the MIMO capacity with PAPC [7].

In this paper we propose an efficient numerical method to solve (7) by manipulating its KKT conditions directly, rather than using the Lagrangian duality method as done previously in [7], [13]. To lighten the notation, the subscript n is to be dropped in the sequel. Let us define $\lambda_i + \lambda_0 = w_i$ for $i = 1, 2, \dots, N$. Then the above problem is equivalent to

$$\underset{w_i,\lambda_0\geq 0}{\text{minimize}} \quad \sum\nolimits_{i=1}^{N}\left(\phi_i w_i - \log w_i\right) \tag{9a}$$

$$\text{subject to} \quad \sum\nolimits_{i=1}^{N} P_i w_i + \tilde{P}_0\lambda_0 = P \tag{9b}$$

$$w_i - \lambda_0 \geq 0,\ i = 1, 2, \dots, N \tag{9c}$$

where $\tilde{P}_0 = P_0 - \hat{P} < 0$. The optimal solution for the above problem can be found by solving the KKT conditions which are given by

$$\mu_i(w_i - \lambda_0) = 0 \tag{10a}$$

$$\mu_0\lambda_0 = 0 \tag{10b}$$

$$\phi_i - 1/w_i + \gamma P_i - \mu_i = 0,\ i = 1, 2, \dots, N \tag{10c}$$

$$\gamma\tilde{P}_0 - \mu_0 + \sum\nolimits_{i=1}^{N}\mu_i = 0 \tag{10d}$$

where $\mu_i \geq 0$ and γ are the KKT multipliers for the constraints $w_i - \lambda_0 \geq 0$ and $\sum_{i=1}^{N} P_i w_i + \tilde{P}_0\lambda_0 = P$, respectively. Note that μ_0 is the Lagrangian multiplier associated with λ_0. We have the following proposition.

Proposition 1. *The solution to the KKT conditions in* (10) *satisfies* $\lambda_0 > 0$.

Proof. See Appendix A. □

Intuitively, the above proposition implies that the SPC is active, which is not surprising. Indeed, the same observation was also made in [14], [16], [17]. It is now obvious that $\mu_0 = 0$. For a given γ, without loss of generality, we can assume that $\frac{1}{\phi_1+\gamma P_1} \geq \frac{1}{\phi_2+\gamma P_2} \geq \dots \geq \frac{1}{\phi_N+\gamma P_N}$. Further, we note that if $\mu_i = 0$ then

$$w_i = 1/(\phi_i + \gamma P_i) \tag{11}$$

Algorithm 1: The Proposed Algorithm for Solving (9).

Input: $\gamma_{\min} = 0, \gamma_{\max}, \epsilon > 0$. Set $n := 0$

repeat

  Set $\bar{\gamma} = \frac{\gamma_{max} - \gamma_{min}}{2}$.

  Rearrange the terms $\{\frac{1}{\phi_i + \bar{\gamma} P_i}\}$ in decreasing order.

  Find the largest $k \leq N-1$ such that $\frac{1}{\phi_k + \bar{\gamma} P_k} \geq \frac{N-k}{\sum_{i=k+1}^{N} \phi_i + \bar{\gamma}(\tilde{P}_0 + \sum_{i=k+1}^{N} P_i)}$.

  Set $w_i = \frac{1}{\phi_i + \bar{\gamma} P_i}$ for $i \leq k$ and $w_i = \lambda_0 = \frac{N-k}{\sum_{i=k+1}^{N} \phi_i + \bar{\gamma}(\tilde{P}_0 + \sum_{i=k+1}^{N} P_i)}$ for $i > k$.

  If $\sum_{i=1}^{N} P_i w_i + \tilde{P}_0 \lambda_0 - P > 0$ then set $\gamma_{\min} = \bar{\gamma}$, otherwise set $\gamma_{\max} = \bar{\gamma}$.

  $n := n + 1$.

until $\gamma_{\max} - \gamma_{\min} < \epsilon$;

Output: $\{w_i\}_{i=1}^{N}$.

which leads to the following proposition.

Proposition 2. *If $i < j$, $w_i > \lambda_0$ and $w_j > \lambda_0$, then $w_i \geq w_j$.*

Proof. It can be easily seen that $\mu_i = \mu_j = 0$; thus $w_i \geq w_j$ follows immediately from (11). □

Proposition 3. *If $w_j = \lambda_0$ for some j, then $w_k = \lambda_0$ for $k > j$.*

Proof. Suppose to the contrary that $w_k > \lambda_0$ for a certain $k > j$. Then $\mu_k = 0$ and thus

$$w_k = \frac{1}{\phi_k + \gamma P_k} \leq \frac{1}{\phi_j + \gamma P_j} \leq \frac{1}{\phi_j + \gamma P_j - \mu_j} = w_j = \lambda_0 \quad (12)$$

which contradicts the assumption that $w_k > \lambda_0$. □

From the two above propositions, we can conclude that there exists a number k such that

$$w_1 \geq w_2 \geq \cdots \geq w_k \geq \lambda_0 \quad (13)$$

$$w_{k+1} = \cdots = w_N = \lambda_0. \quad (14)$$

Using (10c) and (10d), we have

$$\gamma = -\frac{\sum_{i=k+1}^{N} \mu_i}{\tilde{P}_0} = -\frac{\sum_{i=k+1}^{N} \phi_i - \frac{1}{\lambda_0} + \gamma \sum_{i=k+1}^{N} P_i}{\tilde{P}_0} \quad (15)$$

and thus

$$\lambda_0 = \frac{N-k}{\sum_{i=k+1}^{N} \phi_i + \gamma(\tilde{P}_0 + \sum_{i=k+1}^{N} P_i)}. \quad (16)$$

From the above derivations, we propose a bisection method to solve (9) as described in **Algorithm 1**. A possible value of $\gamma_{\max}$ can be chosen as $\gamma_{\max} = \frac{N}{P} - \frac{\phi_{\min}}{P_{\max}}$. A proof that this choice is sufficient to solve (9) is given in Appendix B.

B. Convergence Analysis

First we note that $\mathbf{\Lambda}_n \succ \mathbf{0}$ for all n and thus the proposed CCP-like algorithm is well defined for all iterations. We also assume that the norm of each column of $\mathbf{H}$ is strictly positive, which is usually the case in practice. Otherwise, the corresponding transmit antenna can be removed to obtain a reduced system. In fact, the convergence proof of the proposed CCP-like algorithm follows the same arguments used in [7], and thus we refer the interested reader to [7, Appendix B] for the details.

C. Complexity Analysis

The main operations significantly contributing to the overall complexity of the proposed algorithm include: i) The singular value decomposition (SVD) of $\mathbf{\Lambda}_n^{-1/2}\mathbf{H}^\dagger$, i.e., $\mathbf{\Lambda}_n^{-1/2}\mathbf{H}^\dagger = \mathbf{U}_n \mathbf{\Sigma}_n \mathbf{V}_n^\dagger$, to find $\bar{\mathbf{X}}_n$ (cf. (5)). Note that $\bar{\mathbf{X}}_n$ is expressed as $\bar{\mathbf{X}}_n = \mathbf{V}\tilde{\mathbf{\Sigma}}_n \mathbf{V}^\dagger$ where $\tilde{\mathbf{\Sigma}}$ is diagonal and can be found using the water-filling algorithm [1]; and ii) The inverse of $\mathbf{\Phi}_n$ to find $\mathbf{\Lambda}_{n+1}$. In fact we can easily write $\mathbf{\Phi}_n^{-1} = \left(\mathbf{\Lambda}_n + \mathbf{H}^\dagger \bar{\mathbf{X}}_n \mathbf{H}\right)^{-1} = \mathbf{\Lambda}_n^{-1/2}(\mathbf{I} + \mathbf{U}_n \bar{\mathbf{\Sigma}}_n \mathbf{U}_n^\dagger)^{-1} \mathbf{\Lambda}_n^{-1/2} = \mathbf{\Lambda}_n^{-1/2} \mathbf{U}_n (\mathbf{I} + \bar{\mathbf{\Sigma}}_n)^{-1} \mathbf{U}^\dagger{}_n \mathbf{\Lambda}_n^{-1/2}$, where $\bar{\mathbf{\Sigma}}_n = \mathbf{\Sigma}_n \tilde{\mathbf{\Sigma}}_n$. Since $\mathbf{I} + \bar{\mathbf{\Sigma}}_n$ and $\mathbf{\Lambda}_n$ are both diagonal, the inverse of $\mathbf{\Phi}_n$ is computed efficiently without incurring significant complexity. Hence, the overall complexity of the proposed algorithm is dominated by the complexity of the SVD, which is $4N^2M + 8NM^2 + 9M^3$ [7]. As a result, the complexity of the proposed algorithm is $\mathcal{O}(N^2M)$, which is significantly lower than that of the interior-point solvers i.e., $\mathcal{O}(N^6)$ [7], [20].

IV. Numerical Results

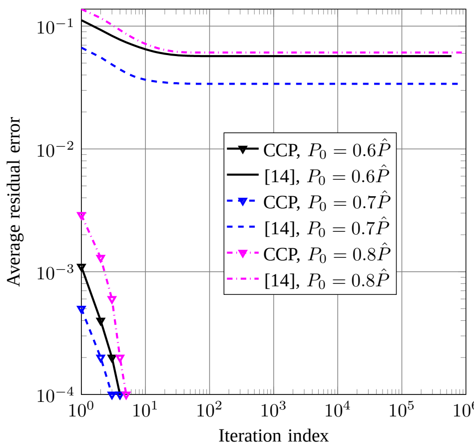
In this section, we evaluate the performance of the proposed algorithm through numerical experiments. Unless explicitly stated otherwise, we consider here the most common case encountered in practice, where each transmit antenna is subject to the same power constraint, i.e., $P_i = \frac{\hat{P}}{N}$, for $i = 1, 2, \ldots, N$. Other relevant simulation parameters are specified for each setup. Note that each averaged result is based on Monte Carlo simulations over 1000 i.i.d. channel realizations. Each entry of the channel matrix is drawn from a circularly-symmetric complex Gaussian distribution with unit variance. Note that we set the initial value of the multiplier $\boldsymbol{\lambda}_0$ associated with the SPC of Algorithm 1 to 1 so that Proposition 1 always holds. The MATLAB code was executed on a 64-bit desktop that supports 32 GB RAM and Intel Xeon Gold.

In the first experiment, we study the convergence performance of the proposed algorithm for different settings of the power constraints. For the purpose of comparison, we also include the projected factored gradient descent method recently proposed in [14], which is a suboptimal iterative method. The residual error is defined as the absolute difference between the objective and the channel capacity computed using CVX [21]. As can be observed in Fig. 1, for the above-described random channel model our proposed algorithm takes less than ten iterations to reach an average error of 10^{-4} and is less sensitive to different settings of SPC and PAPC. In contrast, the convergence of the PFGD algorithm in [14] is very slow and sensitive to different power settings. We also find that it is very difficult to tune the step size of the PFGD algorithm to achieve satisfactory convergence performance.

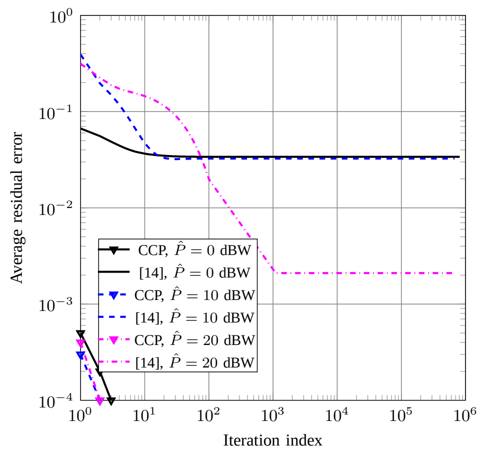
Next, we show that the closed-form solutions in [16], [17] cannot obtain the optimal solution in general. To this end, let us consider the MIMO channel given by

$$\mathbf{H} = \begin{bmatrix} 0.1189 + 0.1515i & 0.1238 + 0.3326i & 0.8572 + 0.1131i \\ -0.3198 - 0.3663i & -0.6491 + 0.2784i & 0.3392 - 0.1974i \\ -0.1019 + 0.6639i & 0.3663 - 0.3097i & -0.1116 - 0.1101i \end{bmatrix}$$

and compare its capacity under joint SPC and PAPC using the proposed approach and that of [16], [17]. Note that each entry of this MIMO channel was randomly generated following a circularly-symmetric complex Gaussian distribution with unit



(a) Varying power constraint ratio, $\hat{P} = 0$ dBW



(b) Varying PAPC, $P_0 = 0.7\hat{P}$

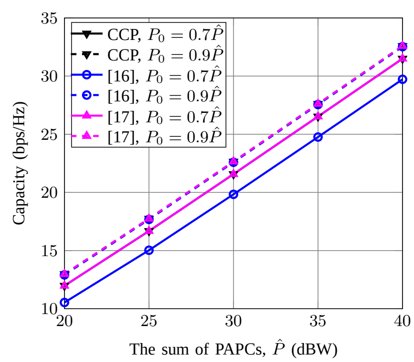
Fig. 1: Convergence rate of the proposed algorithm compared to that of [14] under joint SPC and PAPC with $N = 4$ transmit antennas and $M = 2$ receive antennas.

variance. In order to execute the solution in [17], we need to set the power of of all transmit antennas to be equal. Also, the closed-form solution in [17] requires that the total transmit power must be $P_0 \geq 8.3137$ W and $\hat{P} \geq 15.3213$ W. In other words, the SNR should be larger than 9.198 dB for this method to work. In low power scenarios, the approaches in [16], [17] yield an input covariance matrix that violates the positive semidefinite constraints. To achieve a fair comparison, we compensate for negative eigenvalues and scale the resulting covariance matrix by an appropriate factor so that all power constraints are met. As can be seen from Fig. 2a, the capacity generated by our algorithm and that of [17] are the same in the high power constraint regime, as expected. However, in the low power regime, our proposed solution outperforms other methods (c.f. Fig. 2b). More importantly, our solution is applicable to the general cases of MIMO without imposing any special conditions either on channels or power constraints.

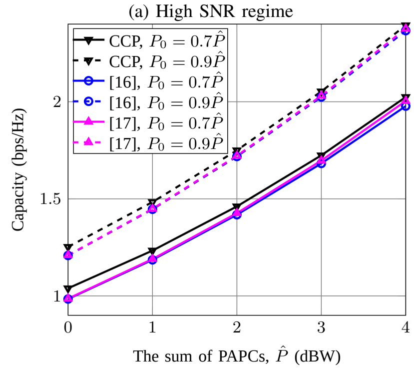
For the purpose of benchmarking, we report in Table I the average run time of our approach along with that of common interior-point solvers i.e., SEDUMI [22], SDPT3 [23] and MOSEK, which generate the optimal solutions without imposing stringent conditions on power constraints and/or channels as in [16], [17]. The solvers are executed through the parser CVX [21]. The ratio $P_0/\hat{P}$ and the error tolerance for the proposed algorithm are set to 0.8 and 10^{-5}, respectively. Note that the run time accounts for both the number of iterations and the per-iteration complexity. We recall that the per-iteration complexity of an interior-point-based method for this problem is $\mathcal{O}(N^6)$ [20], compared to $\mathcal{O}(N^2M)$ for our proposed algorithm. In addition, as illustrated in Fig. 1, the proposed algorithm converges very fast. Thanks to these two properties, the proposed algorithm consistently shows a low run time, which is relatively independent of $\hat{P}$, as can be seen clearly from Table I. We can also see that interior-point-based convex solvers are not suitable for large-scale MIMO systems because their complexity and memory requirements can increase rapidly with the problem size, resulting in prohibitive computation time.

TABLE I: Comparison of the average run time (in seconds) of the proposed algorithm and that of standard solvers, where $P_0 = 0.8\hat{P}$, $M = 2$ receive antennas. The run time is averaged over 1000 channel realizations.

$\hat{P}$	Algorithms/solvers	No. of transmit antennas N			
		16	32	64	128
0 dBW	CCP	**0.0607**	**0.1361**	**0.4244**	**2.1313**
	MOSEK	0.5236	0.5550	1.3228	9.6038
	SEDUMI	0.6524	1.1660	3.6720	18.1569
	SDPT3	1.0664	2.2465	11.5979	71.8263
10 dBW	CCP	**0.0552**	**0.1322**	**0.4472**	**2.2777**
	MOSEK	0.4428	0.5558	1.3350	9.6343
	SEDUMI	0.6702	1.2379	3.8908	19.0718
	SDPT3	1.1308	2.4439	11.1359	73.1988



(a) High SNR regime



(b) Low SNR regime

Fig. 2: Capacity comparison of the proposed algorithm and existing methods [16], [17] under joint SPC and PAPC with $M = 3$ receive antennas and $N = 3$ transmit antennas.

V. Conclusion

We have proposed a low-complexity approach to computing the general case of MIMO capacity with joint SPC and PAPC. By transforming the original problem to an equivalent minimax problem, we take advantage of the special structure of the problem to derive analytical solution to the problem of interest. Extensive analytical and numerical results have demonstrated that our solution is not only low-complexity, fast-converging but also more general than existing methods.

Appendix

A. Proof of Proposition 1

Suppose to the contrary that $\lambda_0 = 0$. Then it immediately holds that $\mu_i = 0$ and $w_i = \frac{1}{\phi_i + \gamma P_i}$ for $i = 1, 2, \ldots, N$. From (10d) we have

$$\gamma = \mu_0 / \tilde{P}_0 \leq 0. \tag{17}$$

As a result, the following inequality is obtained:

$$\sum_{i=1}^{N} P_i w_i = \sum_{i=1}^{N} \frac{P_i}{\phi_i + \gamma P_i} \geq \sum_{i=1}^{N} \frac{P_i}{\phi_i} \geq \sum_{i=1}^{N} P_i(\lambda_{n,i} + \lambda_{n,0}), \tag{18}$$

where the superscript denotes the iteration index and the last inequality is due to the fact that $\phi_i = \left[\left(\mathbf{\Lambda}_n + \mathbf{H}^{\dagger}\bar{\mathbf{X}}_n\mathbf{H}\right)^{-1}\right]_{i,i} \leq \left(\lambda_{n,i} + \lambda_{n,0}\right)^{-1}$. Note that $\lambda_{n,i}$ and $\lambda_{n,0}$ are a solution to (7), and thus

$$\sum_{i=1}^{N} P_i(\lambda_{n,i} + \lambda_{n,0}) = \underbrace{\sum_{i=1}^{N+1} P_i \lambda_{n,i}}_{P} + \underbrace{\Big(\sum_{i=1}^{N} P_i - P_0\Big)}_{>0} \lambda_{n,0} > P. \tag{19}$$

It is easy to see that if the initial value $\lambda_{m,0}$, for $m = 0$ is chosen to be strictly positive, then $\lambda_{m,0} > 0$ for $n \geq m \geq 1$ by induction. Combining (18) and (19) yields

$$\sum\nolimits_{i=1}^{N} P_i w_i > P \tag{20}$$

which indicates that w_i's are not feasible to (9) and thus completes the proof.

B. A Possible Choice of γ_{max} in the Bisection Search

First note that (10c) and (10d) produce

$$\phi_i w_i - 1 + \gamma P_i w_i - \mu_i w_i = 0, \; i = 1, 2, \ldots, N \tag{21}$$

$$\gamma \tilde{P}_0 \lambda_0 - \underbrace{\mu_0 \lambda_0}_{0} + \lambda_0 \sum_{i=1}^{N} \mu_i = 0, \tag{22}$$

respectively. From (21), (22), and (10a), we have

$$\sum\nolimits_{i=1}^{N} \phi_i w_i - N + \gamma\Big(\sum\nolimits_{i=1}^{N} P_i w_i + \tilde{P}_0 \lambda_0\Big) = 0, \tag{23}$$

which is equivalent to

$$\sum\nolimits_{i=1}^{N} \phi_i w_i - N + \gamma P = 0 \tag{24}$$

and thus

$$\gamma P \leq N - \phi_{\min} \sum\nolimits_{i=1}^{N} w_i \tag{25}$$

where $\phi_{\min} = \min_{1 \leq i \leq N} \{\phi_i\}$. It is easy to see that

$$P_{\max} \sum\nolimits_{i=1}^{N} w_i \geq \sum\nolimits_{i=1}^{N} P_i w_i = P - \tilde{P}_0 \lambda_0 \geq P \tag{26}$$

where $P_{\max} = \max_{1 \leq i \leq N} \{P_i\}$. Combining (25) and (26) yields

$$\gamma \leq N/P - \phi_{\min}/P_{\max}. \tag{27}$$

Thus it is sufficient to set $\gamma_{\max} = N/P - \phi_{\min}/P_{\max}$ for the bisection method to solve (9).

References

[1] E. Telatar, "Capacity of multi-antenna Gaussian channels," *Eur. Trans. Telecommun*, vol. 10, pp. 585–598, Nov. 1999.

[2] M. Vu, "MISO capacity with per-antenna power constraint," *IEEE Trans. Commun.*, vol. 59, no. 5, pp. 1268–1274, May 2011.

[3] ——, "MIMO capacity with per-antenna power constraint," in *Proc. IEEE GLOBECOM*, Dec. 2011, pp. 1 – 5.

[4] W. Yu and T. Lan, "Transmitter optimization for the multi-antenna downlink with per-antenna power constraints," *IEEE Trans. Signal Process.*, vol. 55, no. 6, pp. 2646–2660, Jun. 2007.

[5] T. M. Pham, and R. Farrell, and L.-N. Tran, "Low-complexity approaches for MIMO capacity with per-antenna power constraint," in *Proc. IEEE VTC-Spring*, Jun. 2017, pp. 1–7.

[6] T. M. Pham, and R. Farrell, and J. Dooley, and E. Dutkiewicz, and D. N. Nguyen, and L.-N. Tran, "Efficient zero-forcing precoder design for weighted sum-rate maximization with per-antenna power constraint," *IEEE Trans. Veh. Technol.*, vol. 67, no. 4, pp. 3640–3645, Apr. 2018.

[7] T. M. Pham, and R. Farrell, and L.-N. Tran, "Revisiting the MIMO capacity with per-antenna power constraint: Fixed-point iteration and alternating optimization," *IEEE Trans. Wireless Commun.*, vol. 18, no. 1, pp. 388–401, Jan 2019.

[8] G. J. Foschini and M. J. Gans, "On limits of wireless communications in a fading environment when using multiple antennas," *Wireless Pers.Commun*, vol. 6, pp. 311–335, Mar. 1998.

[9] L. Zhang, R. Zhang, Y. C. Liang, Y. Xin, and H. V. Poor, "On Gaussian MIMO BC-MAC duality with multiple transmit covariance constraints," *IEEE Trans. Inf. Theory*, vol. 58, no. 4, pp. 2064 – 2078, Apr. 2012.

[10] H. Huh, H. C. Papadopoulos, and G. Caire, "Multiuser MISO transmitter optimization for intercell interference mitigation," *IEEE Trans. Signal Process.*, vol. 58, no. 8, pp. 4272 – 4285, Aug. 2010.

[11] Y. Yang, G. Scutari, P. Song, and D. P. Palomar, "Robust MIMO cognitive radio systems under interference temperature constraints," *IEEE J. Sel. Areas Commun.*, vol. 31, no. 11, pp. 2465–2482, Nov. 2013.

[12] T. M. Pham, R. Farrell, H. Claussen, M. F. Flanagan, and L.-N. Tran, "Weighted sum rate maximization for zero-forcing methods with general linear covariance constraints," in *Proc. IEEE ICC*, May 2018, pp. 1–6.

[13] ——, "On the MIMO capacity with multiple linear transmit covariance constraints," in *Proc. IEEE VTC-Spring*, Jun. 2018, pp. 1–6.

[14] R. Chaluvadi, S. S. Nair, and S. Bhashyam, "Optimal multi-antenna transmission with multiple power constraints," *IEEE Trans. Wireless Commun.*, vol. 18, no. 7, pp. 3382–3394, Jul. 2019.

[15] P. L. Cao, T. J. Oechtering, R. F. Schaefer, and M. Skoglund, "Optimal transmit strategy for MISO channels with joint sum and per-antenna power constraints," *IEEE Trans. Signal Process.*, vol. 64, no. 16, pp. 4296 – 4306, Aug. 2016.

[16] P. L. Cao and T. J. Oechtering, "Optimal transmit strategy for MIMO channels with joint sum and per-antenna power constraints," in *Proc. IEEE ICASSP*, Mar. 2017, pp. 3569–3573.

[17] S. Loyka, "On the capacity of Gaussian MIMO channels under the joint power constraints," *IEEE Commun. Lett.*, vol. 8, no. 2, pp. 332–335, Apr. 2019.

[18] S. Loyka, "The capacity of Gaussian MIMO channels under total and per-antenna power constraints," *IEEE Trans. Commun.*, vol. 65, no. 3, pp. 1035 – 1043, Mar. 2017.

[19] C. Xing, Z. Fei, Y. Zhou, and Z. Pan, "Matrix-field water-filling architecture for mimo transceiver designs with mixed power constraints," in *Proc. IEEE PIMRC*, Aug. 2015, pp. 392–396.

[20] S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge University Press, 2004.

[21] M. Grant and S. Boyd, "CVX: Matlab Software for Disciplined Convex Programming, version 2.1," March 2014.

[22] J. F. Sturm, "Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones," *Optimization Methods and Software*, vol. 11–12, pp. 625–653, 1999.

[23] K. C. Toh, M. J. Todd, and R. Tutuncu, "SDPT3— a Matlab software package for semidefinite programming," *Optimization Methods and Software*, Nov. 1999.