# On Characterizing the Capacity Region of Massive MIMO Systems with Joint Power Constraints

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***Abstract*—In this paper we consider the problem of computing the capacity of multi-user Gaussian MIMO systems under multiple linear transmit covariance constraints (LTCCs). These LTCCs are general enough to include many transmit power constraints such as sum power constraint (SPC) or per-antenna power constraint (PAPC) as special cases. For the considered MIMO systems with multiple LTCCs, existing solutions are based on subgradient or gradient descent methods, which are known to have slow convergence in general and are therefore not applicable to massive MIMO systems. In contrast, we propose a low-complexity semi-closed-form approach to computing the MIMO capacity for the system of interest. To this end, the considered problem in the broadcast channel is transformed into an equivalent minimax problem in the multiple access channel. The special structure of the minimax problem allows us to derive water-filling-like algorithms based on a novel combination of alternating optimization and concave-convex procedure. For the important case of joint SPC and PAPC, we also propose analytical expressions to find the optimal covariance matrix. Extensive analytical and numerical results are provided to demonstrate the effectiveness of our approach under various massive MIMO system settings.**

***Index Terms*—massive MIMO, linear transmit covariance constraints, sum power constraint, per-antenna power constraint, minimax duality, concave-convex procedure.**

## I. Introduction

The capacity of a Gaussian multiple-input multiple-output (MIMO) channel is commonly investigated under either a sum power constraint (SPC) or per-antenna power constraint (PAPC) [1]–[8]. In practice, a system can be not only subject to a power budget i..e, a SPC, but also a PAPC in order to satisfy the linearity requirements of the power. In cognitive radio networks, we can also impose interference temperature constraints on a secondary user (SU) to limit the interference generated at a primary user (PU) [9]–[11]. Therefore, the research on joint power constraints [9] is of practical relevance and an important issue in MIMO systems.

In general, all of the constraints above can be modeled as linear transmit covariance constraints (LTCCs) [9]. In recent years, there has been growing interest in the general case of LTCCs [12]–[18]. In fact, the majority of these research works investigate single-user MIMO (SU-MIMO) under the special case of joint SPC and PAPC in which either closed-form or algorithmic solutions can be efficiently derived. The determination of the capacity and optimal transmit covariance matrices for both Gaussian SU- and multi-user MIMO (MU-MIMO) channels subject to the general form of LTCCs have remained relatively open problems. For MU-MIMO systems, preliminary studies utilized interior-point and subgradient methods to compute optimal transmit covariance matrices for dirty paper coding (DPC) [9] and zero-forcing (ZF) [10] approaches. However, these methods do not scale favorably with the system size. In fact, it was demonstrated in [19] that these high-complexity methods are not useful for massive MIMO systems. To the best of the authors' knowledge, only [16] has proposed an efficient low-complexity approach relying on alternating optimization (AO) and convex-concave procedure (CCP) to solve the problem of maximizing the MU-MIMO systems with zero-forcing under multiple power constraints.

In this paper we propose an approach to computing the capacity of an MU-MIMO system with DPC under multiple LTCCs in general, and under joint SPC and PAPC in particular. More specifically, we show that the approach in [16] is also applicable to the problem of interest. First, the considered problem in the broadcast channel (BC) is transformed into an equivalent minimax problem in the dual multiple access channel (MAC), generalizing several results on the BC-MAC duality in the previous studies of [7], [9], [20]. The problem then boils down to finding a saddle point of the equivalent minimax formulation. Towards this end, we combine alternating optimization (AO) and concave-convex procedure (CCP) to arrive at an iterative algorithm, where each iteration is based on closed-form expressions. More importantly, we propose a semi-closed-form solution to the important case of joint SPC and PAPC, which has been never reported before. Our contributions are summarized as follows:

- For the general case, we express the capacity of the BC with multiple LTCCs as a minimax optimization problem in the dual MAC, utilizing several results regarding BC-MAC duality. The objective of the minimax problem is a concave-convex function of transmit and noise covariance matrices, respectively.
- We then propose a low-complexity approach to computing a saddle point of the minimax problem by efficiently combining AO and CCP. The idea is to alternately optimize the transmit and noise covariance matrices following

the general methodology of AO. For minimax problems, the convergence of a pure AO is not guaranteed in general [7], [21]. The novelty of our proposed method is to optimize a bound of the objective obtained from the CCP when optimizing the noise covariance matrix. The proposed approach is also based on closed-form expressions, and thus outperforms known solutions relying on either subgradient or interior-point methods in [9], [10] in terms of complexity.
- For the special case of MU-MIMO capacity under joint SPC and PAPC, we derive a closed-form solution to computing the noise covariance matrix. In fact, this closed-form solution has not been previously reported in the literature.
- We provide numerical results on the capacity of massive MU-MIMO systems with joint SPC and PAPC and different precoding methods which have not been reported previously.

The remainder of the paper is organized as follows. We present the system model of MU-MIMO with multiple LTCCs in Section II. Section III provides an algorithm to solve this general case. In Section IV we derive closed-form expressions for the special case of joint SPC and PAPC. We present the numerical results in Section V and conclude the paper in Section VI.

*Notation*: Standard notations are used in this paper. Bold lower and upper case letters represent vectors and matrices, respectively. $\mathbf{I}_N$ defines an identity matrix of size $N$; $\mathbf{I}$ and $\mathbf{0}$ define identity and zero matrices respectively, of which the size can be easily inferred from the context. $\mathbb{C}^{M\times N}$ denotes the space of $M\times N$ complex matrices; $\text{tr}(\mathbf{H})$ denotes the trace of $\mathbf{H}$; $\mathbf{H}^\dagger$ and $\mathbf{H}^T$ are Hermitian and ordinary transpose of $\mathbf{H}$, respectively. Furthermore, we denote the expected value of a random variable by $\mathbb{E}[.]$, and $[x]_+=\max(x,0)$. The $i$th unit vector (i.e., its $i$-th entry equal to one and all other entries equal to zero) is denoted by $\mathbf{e}_i$.

## II. System Model

Consider a $K$-user MIMO BC where the base station and each user $k=1,2,\ldots,K$ are equipped with $N$ and $M_k$ antennas, respectively. Let $\mathbf{H}_k$ denote the channel matrix for user $k$, and let $\mathbf{s}$ denote the composite signal that combines the data for all users in the downlink. Then, we can express the received signal at user $k$ as

$$\mathbf{y}_k=\mathbf{H}_k\mathbf{s}+\mathbf{z}_k \tag{1}$$

where $\mathbf{z}_k$ is the Gaussian noise with distribution $\mathcal{CN}(\mathbf{0},\mathbf{I}_M)$. For Gaussian input, it was proved that dirty paper coding is capacity achieving [22]. The problem of finding the capacity region is usually formulated as a weighted sum rate maximization (WSRMax), which is written as

$$\begin{aligned}\underset{\{\mathbf{S}_k\succeq\mathbf{0}\}}{\text{maximize}}\quad & \textstyle\sum_{k=1}^K w_k\log\frac{|\mathbf{I}+\mathbf{H}_k\sum_{i=1}^k\mathbf{S}_i\mathbf{H}_k^\dagger|}{|\mathbf{I}+\mathbf{H}_k\sum_{i=1}^{k-1}\mathbf{S}_i\mathbf{H}_k^\dagger|}\\ \text{subject to}\quad & \textstyle\sum_{k=1}^K \text{tr}(\mathbf{E}_{ik}\mathbf{S}_k)\le P_i,\ \forall i\end{aligned} \tag{2}$$

where $\mathbf{E}_{ik}$ and $\mathbf{S}_k$ are the $i$th predefined positive semidefinite matrix and input covariance matrix for the $k$th user, $P_i$ is the $i$th power constraint, and $w_k$ is the weighting factor assigned to user $k$. Note that each $\mathbf{E}_{ik}$ represents a general linear constraint on the transmit covariance and it can include several types of transmit power constraints as special cases. Some examples are given below:

- If, for some $i$, $\mathbf{E}_i$ is an identity matrix, the resulting constraint becomes $\text{tr}(\mathbf{S})\le P_i$, representing a SPC.
- If, for some $i$, $\mathbf{E}_i=\text{diag}(\mathbf{e}_i)$, the constraint reduces to $[\mathbf{S}]_{i,i}\le P_i$, denoting a maximum power constraint on the $i$th antenna. In this paper, a PAPC means imposing this constraint for all antennas.
- If, for some $i$, $\mathbf{E}_i=\mathbf{G}^\dagger\mathbf{G}$, where $\mathbf{G}$ is the effective channel between a SU and a PU, then the resulting constraint limits the overall interference experienced by the PU [23], [24].

Without loss of generality, we assume in the rest of the paper that $0<w_1\le w_2\le\ldots\le w_K$ and $\sum_{k=1}^K w_k=1$. We remark that the work of [22] proved that the capacity region given in (2) is achievable in the case of PAPC, i.e., in the case where $\mathbf{E}_{ik}=\text{diag}(\mathbf{e}_i)$ for $k=1,2,\ldots K$. Following similar arguments, it can also be shown that the capacity region for the case of multiple LTCCs is also achievable.

## III. Algorithm Description

In this section, we extend the minimax duality approach in [16] and [21], to find the capacity region of the Gaussian MIMO BC with multiple LTCCs. In fact, some approaches relying on either subgradient or interior-point methods have been proposed for this problem [9], [10]. However, these methods were only applicable to small-scale MIMO or MISO because their computational complexity is not appealing for large-scale scenarios such as massive MIMO. Herein, we propose an efficient solution to this problem, which follows the same idea as that of [16], and in which each iteration is based on closed-form expressions.

Denote by $\mathbf{q}=[q_1,q_2,\ldots,q_L]^T$ the vector of the Lagrange multipliers for the power constraints and let $\mathbf{p}=[P_1,P_2,\ldots,P_L]^T$ be the corresponding power constraints. Extending the result of minimax duality in [16], we can equivalently rewrite (2) as

$$\begin{aligned}\min_{\mathbf{q}\ge\mathbf{0}}\ \max_{\{\bar{\mathbf{S}}_k\succeq\mathbf{0}\}}\quad & \textstyle\sum_{k=1}^K\Delta_k\log|\mathbf{Q}_k+\sum_{i=k}^K\mathbf{H}_i^\dagger\bar{\mathbf{S}}_i\mathbf{H}_i|\\ & -w_K\log|\mathbf{Q}_k|\triangleq f(\mathbf{q},\{\bar{\mathbf{S}}_k\})\\ \text{subject to}\quad & \textstyle\sum_{k=1}^K\text{tr}(\bar{\mathbf{S}}_k)=P;\mathbf{p}^T\mathbf{q}=P\end{aligned} \tag{3}$$

where $\Delta_k=w_k-w_{k-1}\ge 0$, $\mathbf{Q}_k=\sum_i q_i\mathbf{E}_{ik}, P=\sum_{i=1}^L P_i$, $\{\bar{\mathbf{S}}_k\}$ and $\{\mathbf{Q}_k\}$ are considered as input covariance and noise covariance matrices in the dual MAC, respectively. Note that when we only consider a PAPC, the above formulation reduces to the one in [20] and [21]. We also note that the objective in (3) is convex with $\mathbf{q}\ge\mathbf{0}$ and concave with $\{\bar{\mathbf{S}}_k\succeq\mathbf{0}\}$. Thus, there is a saddle point for (3). Let $(\mathbf{q}^*,\{\bar{\mathbf{S}}_k^*\})$ be the saddle point of (3). Then the optimal covariance matrices that achieve the capacity region in the BC are given by

$$\mathbf{S}_k^* = \mathbf{M}_k^{-1/2}\mathbf{U}_k\mathbf{V}_k^\dagger\mathbf{B}_k^{1/2}\bar{\mathbf{S}}_k^*\mathbf{B}_k^{1/2}\mathbf{V}_k\mathbf{U}_k^\dagger\mathbf{M}_k^{-1/2} \quad (4)$$

where $\mathbf{U}_k, \mathbf{V}_k$ are achieved from the economy-size SVD of $(\mathbf{M}_k^{-1/2}\mathbf{H}_k^\dagger\mathbf{B}_k^{-1/2})$; $\mathbf{M}_k = \mathbf{Q}_k^* + \sum_{j=k+1}^K \mathbf{H}_j^\dagger\bar{\mathbf{S}}_j^*\mathbf{H}_j, \mathbf{B}_k = \mathbf{I} + \sum_{j=1}^{k-1}\mathbf{H}_k\mathbf{S}_j^*\mathbf{H}_k^\dagger$.

*Remark* 1. We discuss three benefits of using the minimax problem in (3) in computing the capacity region of a Gaussian MIMO BC. Firstly, the $\mathbf{S}_k$-maximization has *the same structure* for all types of LTCCs. Secondly, *the* $\mathbf{q}$*-minimization does not scale with the number of users*. Thirdly, projection onto the feasible sets of $\mathbf{q}$ and $\{\bar{\mathbf{S}}_k\}$ can be done using closed-form expressions. We exploit these properties to derive efficient solutions to the MIMO capacity region.

The proposed method for solving (3) follows the approach in [16], [21], which is described next. Denote $(\mathbf{q}^n, \{\bar{\mathbf{S}}_k^n\})$ as the obtained values of $(\mathbf{q}, \{\bar{\mathbf{S}}_k\})$ after $n$ iterations of the proposed iterative algorithm. For a given $\mathbf{q}^n$, $\bar{\mathbf{S}}^n$ is the solution to the maximization problem under an SPC to which gradient-projection-based methods are numerically shown to be efficient (see [21], [25], [26] for details). Moreover, if the sum capacity is of interest, i.e. $\Delta_k = 0$ for all $k \geq 2$, it is easy to see that the maximization in (3) admits a water-filling solution.

Turning now to the problem of finding $\mathbf{q}^{n+1}$, we solve the optimization problem below:

$$\begin{aligned} \underset{\mathbf{q}\geq\mathbf{0}}{\text{minimize}} \quad & \textstyle\sum_{k=1}^K \Delta_k \log|\mathbf{Q}_k + \sum_{i=k}^K \mathbf{H}_i^\dagger\bar{\mathbf{S}}_i\mathbf{H}_i| \\ & -w_K \log|\mathbf{Q}_k| \\ \text{subject to} \quad & \mathbf{p}^T\mathbf{q} = P. \end{aligned} \quad (5)$$

In light of CCP, we choose to minimize an upper bound of the objective instead of optimizing the original objective in (5). To this end, by invoking the concavity of the logdet function, we obtain the following inequality

$$\log|\mathbf{Q}_k + \textstyle\sum_{i=k}^K \mathbf{H}_i^\dagger\bar{\mathbf{S}}_i^n\mathbf{H}_i| \leq \log|\mathbf{\Phi}_k^n| + \operatorname{tr}\big(\mathbf{\Phi}_k^{-n}\big(\mathbf{Q}_k - \mathbf{Q}_k^n\big)\big) \quad (6)$$

where $\mathbf{\Phi}_k^n = \mathbf{Q}_k + \sum_{i=k}^K \mathbf{H}_i^\dagger\bar{\mathbf{S}}_i^n\mathbf{H}_i, \mathbf{\Phi}_k^{-n} \triangleq (\mathbf{\Phi}_k^n)^{-1}$. Thus, $\mathbf{q}^{n+1}$ is found to be the optimal solution to the following problem

$$\begin{aligned} \underset{\mathbf{q}\geq\mathbf{0}}{\text{minimize}} \quad & \textstyle\sum_{k=1}^K \frac{\Delta_k}{w_K}\operatorname{tr}\Big(\mathbf{\Phi}_k^{-n}\mathbf{Q}_k\Big) - \log|\mathbf{Q}_k| \\ \text{subject to} \quad & \mathbf{p}^T\mathbf{q} = P. \end{aligned} \quad (7)$$

We remark that the problem (7) has a similar form to [17, Eq. (10)]; thus a gradient-projection-based algorithm can be easily customized to apply here. The algorithm description to solve (3) is summarized in Algorithm 1. The convergence proof is similar to those of [16], [17], [21] and thus skipped for the sake of brevity.

## IV. MIMO Capacity Region with Joint SPC and PAPC

In this section we deal with the specific case of the MIMO capacity region with joint SPC and PAPC. We remark that no efficient solutions have been reported for this important case previously. For the SU-MIMO case, it is possible to find closed-form solutions based on solving the KKT conditions for some specific scenarios as shown in [15], [17]. However, such a method appears to be impossible for the MU-MIMO case.

**Algorithm 1:** The Proposed Algorithm for Solving (3).

**Input:** $\mathbf{q}^0$, $\epsilon > 0$.
Initialize $n := 0$, and $\tau = 1 + \epsilon$.
**while** $\tau > \epsilon$ **do**
  Compute $\mathbf{Q}_k^n = \sum_{i=1}^L q_i^n \mathbf{E}_{ik}$.
  Solve $\{\bar{\mathbf{S}}_k^n\} = \underset{\{\bar{\mathbf{S}}_k\succeq\mathbf{0}, \sum_{k=1}^K \operatorname{tr}(\bar{\mathbf{S}}_k)=P\}}{\arg\max} \sum_{k=1}^K \Delta_k \log|\mathbf{Q}_k + \sum_{i=k}^K \mathbf{H}_i^\dagger\bar{\mathbf{S}}_i\mathbf{H}_i|$.
  For $n \geq 1$, compute $\tau = |f(\mathbf{q}^n, \{\bar{\mathbf{S}}_k^n\}) - f(\mathbf{q}^{n-1}, \{\bar{\mathbf{S}}_k^{n-1}\})|$.
  Solve (7) to find $\mathbf{q}^{n+1}$ using [17, Alg. 1].
  $n := n + 1$.
**end**
**Output:** $\{\bar{\mathbf{S}}_k^n\}_{k=1}^K$ and apply (4) to compute optimal $\{\mathbf{S}_k^n\}_{k=1}^K$.

Our main point is to demonstrate that the equivalent minimax formulation in the MAC allows for efficient solutions to this special case. In particular, for the case of joint SPC and PAPC considered in this paper, we show that solving (7) admits a closed form-solution.

Notice that the number of constraints is $L = N + 1$ for the considered problem ; $\mathbf{E}_{N+1} = \mathbf{I}_N$ and $\mathbf{E}_i = \operatorname{diag}(\mathbf{e}_i)$ for $i = 1, 2, \ldots, N$ are associated with the SPC and PAPC, respectively. As a consequence, $P_i (i = 1, 2, , \ldots, N)$ is the power constraint for the individual antenna and $P_{N+1} \triangleq P_T$ is the power budget. In the following, we only consider the non-trivial case in which $\min\{P_i\} < P_{N+1} < \sum_{i=1}^N P_i$.

We begin by rewriting (7) as

$$\begin{aligned} \underset{\mathbf{q}\geq 0}{\text{minimize}} \quad & q_{N+1}\psi_{n,N+1} + \textstyle\sum_{i=1}^N (q_i\psi_{n,i} - \log(q_{N+1} + q_i)) \\ \text{subject to} \quad & \textstyle\sum_{i=1}^{N+1} P_i q_i = P \end{aligned} \quad (8)$$

where $\psi_i = [\sum_{j=1}^K \frac{\Delta_j}{w_K}\mathbf{\Phi}_j^{-n}]_{i,i}$; $q_{N+1}$ and $q_i$ for $i = 1, 2, \ldots, N$ are the Lagrange multipliers for the SPC and PAPC, respectively.

**Theorem 1.** *The solution to* (7) *in the special case of joint SPC and PAPC is given by*

$$q_{N+1} = \frac{N - \bar{k}}{(\psi_{N+1} - \sum_{i=1}^{\bar{k}}\psi_i) + \gamma(P_T - \sum_{i=1}^{\bar{k}} P_i)}, \quad (9)$$

$$q_i = 0, \quad i = \bar{k}+1, \bar{k}+2, \ldots, N \quad (10)$$

$$q_i = \frac{1}{\psi_i + \gamma P_i} - \frac{N - \bar{k}}{(\psi_{N+1} - \sum_{i=1}^{\bar{k}}\psi_i) + \gamma(P_T - \sum_{i=1}^{\bar{k}} P_i)}, \quad i = 1, \ldots, \bar{k} \quad (11)$$

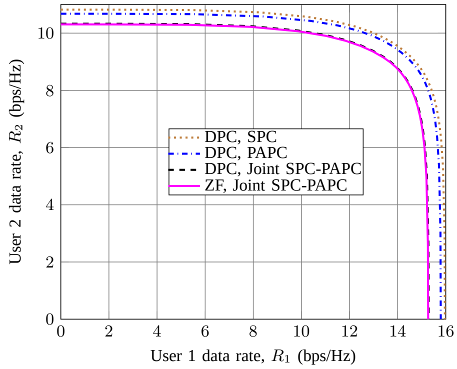


Fig. 1: Comparison of capacity regions with different precoding methods and different power constraints for a massive MIMO system with $N = 128$ transmit antennas, $M = 2$ receive antennas and $K = 2$ users. The sum power constraint is $P_T = 16$ dBW for the SPC case, the sum of PAPCs is equal to $P_A = 16$ dBW for the PAPC case, and for the case of joint SPC and PAPC we set $P_A = 16$ dBW and $P_T = 0.8P_A$.

*where $\bar{k}$ is the largest $\bar{k} \leq N-1$ such that*

$$\frac{1}{\psi_i + \gamma P_i} \geq \frac{N-\bar{k}}{(\psi_{N+1} - \sum_{i=1}^{\bar{k}} \psi_i) + \gamma(P_T - \sum_{i=1}^{\bar{k}} P_i)} \quad (12)$$

*and $\gamma$ is the solution of the equation*

$$\sum_{i=1}^{\bar{k}} \frac{P_i}{\psi_i + \gamma P_i} + \frac{(N-\bar{k})(P_T - \sum_{i=1}^{\bar{k}} P_i)}{(\psi_{N+1} - \sum_{i=1}^{\bar{k}} \psi_i) + \gamma(P_T - \sum_{i=1}^{\bar{k}} P_i)} = P. \quad (13)$$

*Proof.* See the Appendix. □

## V. Numerical Results

In this section, we take advantage of our low-complexity algorithms to study the performance of massive MIMO systems under the important case of joint SPC and PAPC. For notational convenience, we denote the sum of PAPCs as $P_A = \sum_{i=1}^{N} P_i$. Unless explicitly stated otherwise, we consider here the most common case encountered in practice, where each transmit antenna is subject to the same power constraint, i.e., $P_i = P_0 = \frac{P_A}{N}$, for $i = 1, 2, \ldots, N$. As mentioned earlier, we are interested in the nontrivial case where $\min\{P_i\} < P_T < P_A$. In fact, if $\min\{P_i\} \geq P_T$, then the PAPC can be removed without loss of optimality. Similarly, if $P_T \geq P_A$, then the SPC can be eliminated. All users are equipped with the same number of receive antennas, i.e., $M_k = M$. Unless stated otherwise, the error tolerance $\epsilon$ is set to $10^{-6}$ for all simulations. Other relevant simulation parameters are specified for each setup.

In the first experiment, we characterize the capacity region of the optimal nonlinear precoding method (DPC) and the common linear precoding (ZF) [16] in a realistic massive MIMO scenario under joint SPC and PAPC. In particular, we consider the typical urban micro-cell WINNER II B1 channel model [27] where two users are distributed around a centered base station in a single cell. In addition, we only consider the path loss and ignore shadowing. The noise power is set to $-94$ dBm over a bandwidth of 100 MHz. The base station and each user are equipped with 128 and 2 antennas, respectively. We can see clearly from Fig. 1 that the capacity of ZF with joint SPC and PAPC is close to that of DPC in massive MIMO settings. Both are less than the capacity of DPC with PAPC and the feasible region is still bounded by the SPC.

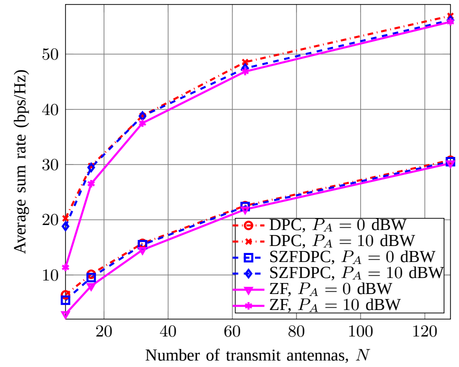


Fig. 2: Average sum rate of different precoding methods i.e., ZF, DPC and SZFDPC under joint SPC and PAPC with $M = 2$ receive antennas, $K = 4$ users. Here the sum power constraint is $P_T = 0.8P_A$.

In Fig. 2 we study the performance of the average sum rate of different precoding methods, including ZF, successive zero-forcing DPC (SZFDPC) [16] and DPC under joint SPC and PAPC. For the same set of power constraints, the average sum rate of ZF is lower than that of suboptimal precoding SZFDPC, while DPC remains the optimal solution with the highest sum rate. We can also see that when the number of transmit antennas increases, the performance of ZF and SZFDPC methods approaches that of DPC.

## VI. Conclusions

We have proposed an efficient approach to computing the MIMO capacity and characterizing the capacity region under an arbitrary combination of linear transmit covariance constraints. The approach is based on minimax duality and CCP to derive water-filling-like algorithms. For the special case of the MIMO capacity with joint SPC and PAPC, we have also provided an analytical solution. In the numerical results, we have, for the first time, studied the performance of massive MIMO systems under joint power constraints as well as different precoding methods. In fact, our solutions help to overcome the computational difficulties of previously published algorithms which mostly rely on subgradient or interior-point methods.

## Acknowledgements

This publication has emanated from research supported in part by a grant from Science Foundation Ireland under Grant number 17/CDA/4786 and in part by a research grant from Science Foundation Ireland (SFI) and is co-funded under the European Regional Development Fund under Grant Number 13/RC/2077.

## APPENDIX

We can write the KKT conditions for the considered problem as

$$\mu_i q_i = 0 \tag{14}$$

$$\psi_i - \frac{1}{q_{N+1} + q_i} + \gamma P_i - \mu_i = 0, i = 1, 2, \ldots, N \tag{15}$$

$$\psi_{N+1} - \sum\nolimits_{i=1}^{N} \frac{1}{q_{N+1} + q_i} + \gamma P_T - \mu_{N+1} = 0, i = N + 1. \tag{16}$$

From (14), if $q_i > 0$ for $i = 1, 2, \ldots, N$ then the corresponding $\mu_i = 0$ which results in

$$\mu_{N+1} = \gamma(P_T - \sum\nolimits_{i=1}^{N} P_i). \tag{17}$$

In this paper, we only consider the case where the sum power constraint is less than the total power of PAPC i.e., $P_T < \sum_{i=1}^{N} P_i$, therefore $\mu_{N+1} = 0, q_{N+1} > 0$.

Without loss of generality, we can sort $\{\frac{1}{\psi_i + \gamma P_i}\}$ in decreasing order. From (15), if $P_i$ is active, we obtain

$$q_{N+1} + q_i = \frac{1}{\psi_i + \gamma P_i}. \tag{18}$$

As a result, we have $q_i > q_j$ if $q_i > 0$ and $q_j > 0$ where $i < j$. In addition, we can easily prove that if $q_j = 0$ for some $j$ then $q_{\bar{k}} = 0$ for $\bar{k} > j$. Hence, we can find an integer $\bar{k}$ such that

$$q_1 \geq q_2 \geq \cdots \geq q_{\bar{k}} > 0 \tag{19}$$

and

$$q_{\bar{k}+1} = q_{\bar{k}+2} = \cdots = q_N = 0. \tag{20}$$

Based on these results, combining (15) and (16) results in

$$q_{N+1} = \frac{N - \bar{k}}{(\psi_{N+1} - \sum_{i=1}^{\bar{k}} \psi_i) + \gamma(P_T - \sum_{i=1}^{\bar{k}} P_i)} \tag{21}$$

$$q_i = \frac{1}{\psi_i + \gamma P_i} - \frac{N - \bar{k}}{(\psi_{N+1} - \sum_{i=1}^{\bar{k}} \psi_i) + \gamma(P_T - \sum_{i=1}^{\bar{k}} P_i)}. \tag{22}$$

Substituting these values of $q_i$ into the power constraint $\sum_{i=1}^{N+1} P_i q_i = P$ , we obtain

$$\sum_{i=1}^{\bar{k}} \frac{P_i}{\psi_i + \gamma P_i} + \frac{(N - \bar{k})(P_T - \sum_{i=1}^{\bar{k}} P_i)}{(\psi_{N+1} - \sum_{i=1}^{\bar{k}} \psi_i) + \gamma(P_T - \sum_{i=1}^{\bar{k}} P_i)} = P \tag{23}$$

whose value of $\gamma$ can be solved easily by the Newton method or bisection method.

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