Efficient Communications for Overlapped Chirp-based Systems

Thuy M. Pham, André N. Barreto, and Gerhard P. Fettweis

***Abstract*—In this paper, we consider the problem of increasing the spectral efficiency of a chirp-based communication system. Existing radar systems employ non-overlapping chirps with a large time-bandwidth product, and a modulation of these chirps for communications systems results in low spectral efficiency. Chirp overlapping can enhance the data rate, but it suffers from performance degradation due to intersymbol interference (ISI). In this study, we derive a useful bit error rate (BER) expression to analyze the performance of a system with a finite number of overlapped chirps. Additionally, we investigate the conditions in which the ISI can be reduced while approaching the Nyquist signaling rate. We then propose simple linear equalization techniques to compensate for the ISI, allowing us to achieve faster-than-Nyquist signaling. We demonstrate the effectiveness of our approach by extensive analytical and numerical results.**

***Index Terms*—chirp overlapping, communications, equalization, spectral efficiency, ISI, BER.**

I. Introduction

Chirp spread spectrum has been developed and deployed extensively for military radar systems dated back to the 1940s, and, more recently, for automotive radar. Additionally, chirp signals can be employed for communications owning to their appealing features, such as potentially high processing gain, low-power implementation and robustness against channel impairments [1]–[6]. In recent years, industry and academia have shown growing interest in coexistence of sensing and communications for applications such as automotive radar and air traffic control [7], [8]. Thus, a chirp-based waveform has a great potential for efficient communication, and possibly joint radar sensing and communication.

There are several approaches to use chirp signals for communications. A straightforward approach is to transmit frequency-shift keying (FSK)-modulated non-overlapping chirps [9], which can still be used for radar detection, but this method results in low spectral efficiency. An overlap technique in either frequency or time domain can increase the spectral efficiency, and, consequently, the data rate. In frequency domain, orthogonal chirp division multiplexing (OCDM) relies on frequency-shifted and -folded chirps to obtain orthogonal signals [6], [10]. Although the spectral efficiency can approach the Nyquist rate, this solution requires fully-digital signal-processing and complex receivers. Another one is to allow chirps to overlap in the time domain to increase the data rate of a system and use a simple matched filter for detection. In this overlap-based system, data can be modulated by either binary orthogonal keying (BOK) or direct modulation (DM). The former utilizes up-chirp and down-chirp for representing data, i.e., zero and one, [2], [3] while the latter employs a chirp as a spreading code [5]. As a result, a transmitter and the corresponding receiver in a DM system only need one chirp as pulse-shaping and various modulation schemes can be therefore utilized in the system.

Despite its simplicity, the overlap technique suffers from intersymbol interference (ISI), which results in performance degradation. In [11] the condition such that evenly-spaced overlapping chirps are orthogonal was described, but a proof is missing. Having orthogonal chirps eliminates ISI in single-path channels, but for commonly employed parameters these conditions result in low spectral efficiency. Therefore, determining an appropriate interval trading off spectral efficiency and ISI is of special interest. In fact, a concrete analysis regarding bit error rate (BER) and an optimal interval is still missing in [2], [3], [6], [9]. To the best of our knowledge, only [5] proposes a theoretical BER expression for an infinite number of overlapped chirps. However, they only investigate transmission at a rate much lower than the Nyquist signaling rate and do not provide any solutions to compensate for the ISI.

In this letter, we also study a communications system of overlapped chirps in the time domain. More specifically, we revisit the BER under a finite number of chirps, which is the case in practice, where frames consisting of a number of chirps are transmitted with guard intervals between them. This approach allows the use of simple block-based linear equalizers (EQ) to mitigate the effects of ISI. Interestingly, with equalization we can *achieve the exact Nyquist signaling rate* while keeping BER close to the non-ISI case, which is hardly achievable using the state-of-the-art approaches. Our main contributions include the following:

- We revise the BER expression for an overlapped chirp-based communications system where the number of chirps is finite. The derivation relies on the approximation of Q-function using series of exponentially decreasing cosine [5], [12].
- We derive the conditions for which overlapping chirps are orthogonal, and show that ISI-free communications is not possible at symbol rates close or equal to Nyquist signaling rate. We then numerically evaluate the conditions in which ISI is minimized.
- We analyze the conditions to achieve the Nyquist signaling rate in light of our BER analysis and propose simple but efficient block-based linear equalization techniques to mitigate the effect of ISI on the system.

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- The numerical results have demonstrated the accuracy of our BER expression and the effectiveness of our proposed equalization. In particular, the performance of a system under severe ISI using linear equalization still approaches its performance without ISI.

Notation: Bold lower and upper case letters represent vectors and matrices, respectively. $\mathbf{I}$ defines an identity matrix, of which the size can be easily inferred from the context; $\mathbf{H}^T$ is transpose of $\mathbf{H}$; $\mathbb{E}[.]$ and $\mathcal{P}(.)$ is the expectation and probability of a random variable. $\circledast$ denotes the convolution, $\Re(.)$ stands for a real value and $\mathbb{Z}$ is the set of integer numbers.

II. System Model

A complex baseband chirp signal is given by

$$s(t) = \sqrt{\frac{E_b}{\tau_d}} e^{j\pi\mu t^2}, \quad |t| \leq \frac{\tau_d}{2} \tag{1}$$

where E_b is the symbol energy and $\mu = \frac{B}{\tau_d}$ is the chirp slope, in which B and τ_d are the bandwidth and the chirp duration, respectively. Hence, a direct modulated chirp can be expressed as

$$u(t) = \xi s(t) \tag{2}$$

where $\xi = \pm 1$ for binary phase shift keying (BPSK). We assume that the receiver is synchronized with the transmitter and the impulse response of the matched filter at the receiver side is defined by

$$h(t) = \sqrt{\tau_d} e^{-j\pi\mu t^2}. \tag{3}$$

Then, we may express the filter output as

$$w(t) = u(t) \circledast h(t) = \xi \sqrt{E_b} r(t), \tag{4}$$

where $r(t)$ is the chirp autocorrelation function defined by [2], [5]

$$r(t) = \frac{\sin\left(\pi B t (1 - \frac{|t|}{\tau_d})\right)}{\pi B t}, \quad |t| \leq \tau_d. \tag{5}$$

To increase the data rate of the system, we may send multiple chirps with a signaling interval $\tau \leq \tau_d$. In words, the transmitted signals are overlapped and an output of the matched filter at the time $t_i = i\tau$ is, therefore, as follows

$$y_i = \sum_{k=-\infty}^{+\infty} w_{i-k} + n_i = \sum_{k=-\infty}^{+\infty} \xi_k \sqrt{E_b} r((i-k)\tau) + n_i, \tag{6}$$

where n_i is complex non-white Gaussian noise with zero-mean and variance $\sigma_n^2 = N_0$.

III. Proposed Approach

A. BER expression for overlapped chirps

We propose to send a frame with a finite number of κ overlapped chirps, preceded or followed by a guard interval. As a consequence, we can simplify the aforementioned equation as follows

$$y_i = \sum_{k=0}^{(\kappa-1)} \xi_k \sqrt{E_b} r((i-k)\tau) + n_i = \sqrt{E_b}\left(\xi_i + v_i\right) + n_i, \tag{7}$$

where v_i is the ISI component

$$v_i = \sum_{k=0, k\neq i}^{(\kappa-1)} \xi_k r_{-(k-i)}. \tag{8}$$

It was mentioned in [11] that zero ISI can be obtained with overlapping chirps, i.e., $v_i = 0$, if we choose $\tau = \sqrt{\frac{\tau_d}{B}}$. However, the condition for its feasibility as well as a proof is missing. Here, in Appendix A, we derive more generic conditions at which zero-ISI can be achieved. However, it relies on very stringent conditions. Also, for commonly used chirp parameters, the interval $\tau = \sqrt{\frac{\tau_d}{B}}$ is much larger than the Nyquist signaling interval $\tau_{\mathrm{N}} = 1/B$, and we have always to tolerate or deal with ISI if we want to achieve high spectral efficiency.

Now we revise the BER formulation to our considered problem. Considering a BPSK system, where $\xi_i = [+1, -1]$, the average error probability is given by

$$\begin{aligned} P_e &= \mathcal{P}(y > 0 | \xi_0 = -1) \\ &= \mathcal{P}(-r_0 + v_0 + n > 0 | \xi_0 = -1). \end{aligned} \tag{9}$$

The conditional probability is defined as

$$P_{e|v_0} = \mathcal{P}(n > (r_0 - v_0)) = \mathcal{F}(r_0 - v_0) \tag{10}$$

where $\mathcal{F}(.)$ is the cumulative distribution function of n_i.

Assuming r_0 and v_0 are both normalized, we obtain

$$P_{e|v_0} = \mathcal{F}(1 - v_0). \tag{11}$$

For BPSK, the above equation is equivalent to

$$P_{e|v_l} = Q\left(\sqrt{\frac{2E_b}{N_0}}(1 - v_l)\right). \tag{12}$$

Proposition 1. *A closed-form BER for a system of κ overlapped chirps is given by*

$$P_e = \Re\left[\sum_{j=0}^{n_T-1} c_j e^{\rho} \mathbb{E}\left(\prod_{k=0, k\neq l}^{(\kappa-1)} \cosh(r((l-k)\tau)\rho)\right)\right] \tag{13}$$

where $\rho = (\lambda_j + i\omega_j)\sqrt{\frac{2E_b}{N_0}}$; n_T, c_j, λ_j, and ω_j are taken from the approximation of Q-function in [5].

A proof of this proposition is given in Appendix B. In particular, we derive the expression from the series of exponentially decreasing cosine [5], [12] since it provides useful and appropriate approximation for mathematical manipulations while retaining sufficient accuracy. In Fig. 1 we plot the absolute errors between the approximated and the actual values of the Q-function. Since [5] demonstrates a better approximation in comparison with that of [12], we utilize the former in the present letter. Note that the curve of [5] is not smooth since the optimization for the approximation is performed in two separate error regimes. When the number of chirps is infinite, the ISI is mostly decided by both $(\vartheta-1)$ precursors and $(\vartheta-1)$ successors, where $\vartheta = \frac{\tau_d}{\tau}$ and our BER expression therefore reduces to the one in [5]. Furthermore, we can easily extend our formula to M-PSK where the modulation order $M \geq 4$ [13]. Specifically, a slight change in the derivation shows that

$$P_e = \Re\left[\frac{2}{\log_2 M} \sum_{j=0}^{n_T-1} c_j e^{\rho} \mathbb{E}\left(\prod_{k=0, k\neq l}^{(\kappa-1)} \cosh(r((l-k)\tau)\beta)\right)\right] \tag{14}$$

where $\beta = (\lambda_j + i\omega_j)\sqrt{\frac{2E_b \log_2 M}{N_0}} \sin(\frac{\pi}{M})$. In fact, the proof follows similar arguments to those of Appendix B and thus is skipped for the sake of brevity.

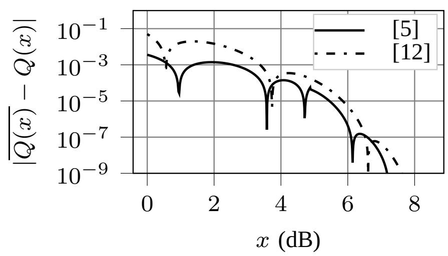

Fig. 1: Performance of different Q-function approximations.

B. Spectral efficiency analysis

For a BPSK system the spectral efficiency is given by

$$\eta = \frac{R}{B} = \frac{1}{\tau B} \tag{15}$$

where R is the bit rate. Hence, a small interval results in high spectral efficiency. Then a problem of interest is how to maximize the spectral efficiency while retaining low BER.

Considering orthogonal transmission with the conditions in Appendix A, we obtain

$$\eta = \frac{R}{B} = \frac{1}{\sqrt{\tau_d B}}. \tag{16}$$

This results in very low spectral efficiency $\eta \ll 1$ since the time-bandwidth product $\tau_d B \gg 1$ as a requirement for radar systems. This requirement makes the spectral efficiency even worse in case of non-overlapping chirps, i.e., $\tau \geq \tau_d$ since

$$\eta \leq \frac{1}{\tau_d B} \ll 1. \tag{17}$$

One possible solution is to transmit consecutive chirps spaced at the first zero of (5), which is given by[1]

$$\tau = \frac{\tau_d - \sqrt{\tau_d^2 - 4\frac{\tau_d}{B}}}{2}. \tag{18}$$

In this case, the spectral efficiency is computed as

$$\eta = \frac{R}{B} = \frac{2}{B\tau_d(1 - \sqrt{1 - \frac{4}{B\tau_d}})} \tag{19}$$

and its limit, as $B\tau_d$ increases, is therefore

$$\lim_{B\tau_d \to \infty} \eta = \lim_{B\tau_d \to \infty} \left(\frac{1}{(1 - \frac{4}{B\tau_d})^{-\frac{3}{2}}} \right) = 1. \tag{20}$$

Generally speaking, the spectral efficiency can approach 1 but the result is hardly hold true in practice where $B\tau_d$ is typically from twenty to several hundreds. Also, this interval only removes the ISI from a single preceding and following chirp, but we still have to deal with the ISI from all the other chirps.

The argument above is also applicable to a Nyquist signaling interval $\tau_N = 1/B$ in which we achieve $\eta = 1$, but cannot eliminate ISI. Note that the approximations in [4] approach the Nyquist rate. However, no techniques are reported to compensate for the ISI in such high spectral efficiency. In contrast, we can utilize simple linear block equalization to mitigate the effect of the ISI as shown in the next subsection. Moreover, we show that this equalization can also work effectively at a faster-than-Nyquist rate.

[1]See Appendix A.

C. Equalization methods

Considering the first received chirp, we can expand (7) as

$$\begin{aligned} y_0 &= \sum_{k=0}^{(\kappa-1)} \xi_k r(-k\tau) + n_0 \\ &= \xi_0 r(0) + \xi_1 r(-\tau) + \dots \\ &\quad + \xi_{\kappa-1} r(-(\kappa-1)\tau) + n_0. \end{aligned} \tag{21}$$

Continue in this fashion, we can obtain similar results for the other chirps. To simplify the formulation, we define an effective channel matrix

$$\mathbf{H} = \begin{bmatrix} r_0 & r_{-1} & \cdots & r_{-(\kappa-1)} \\ r_1 & r_0 & \cdots & r_{-(\kappa-2)} \\ \vdots & \vdots & \ddots & \vdots \\ r_{\kappa-1} & r_{\kappa-2} & \cdots & r_0 \end{bmatrix} \tag{22}$$

where $r_j = r(j\tau)$. Note that $\mathbf{H}$ is symmetric since $r(.)$ is an even function. The received signals can be written as

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n} \tag{23}$$

where $\mathbf{x} = [\xi_0, \xi_1, \dots, \xi_{\kappa-1}]^T$, $\mathbf{y} = [y_0, y_1, \dots, y_{\kappa-1}]^T$, $\mathbf{n} = [n_0, n_1, \dots, n_{\kappa-1}]^T$.

This formulation is quite standard in the literature for which we can apply simple linear equalization methods, for example, zero-forcing (ZF) or minimum mean square error (MMSE). More specifically, the equalization matrices are given by

$$\mathbf{W}_{\mathbf{ZF}} = \mathbf{H}^{-1} \tag{24}$$

$$\mathbf{W}_{\mathbf{MMSE}} = \mathbf{H}^T(\mathbf{H}\mathbf{H}^T + \lambda^{-1}\mathbf{I})^{-1} \tag{25}$$

where λ is the signal to noise ratio.

Since $\mathbf{H}$ is a symmetric matrix, we can compute the equalization matrix efficiently in what follows. By eigenvalue decomposition, we obtain

$$\mathbf{H} = \mathbf{U}\mathbf{\Sigma}\mathbf{U}^T \tag{26}$$

where $\mathbf{U}$ is an orthogonal matrix. Substituting (26) into (24) and (25) yields

$$\mathbf{W}_{ZF} = \mathbf{U}\mathbf{\Sigma}^{-1}\mathbf{U}^T \tag{27}$$

$$\mathbf{W}_{MMSE} = \mathbf{U}\mathbf{\Sigma}(\mathbf{\Sigma}^2 + \lambda^{-1}\mathbf{I})^{-1}\mathbf{U}^T. \tag{28}$$

We notice that, for faster-than-Nyquist rate, i.e., $\tau \to 0$ and thus $r(.) \to 1$ which results in $\mathbf{H} \to \mathbf{J}$ where $\mathbf{J}$ is an all-one-rank-deficient- matrix. Therefore, these methods are projected at higher BER than those of non-overlapping or Nyquist case. The performance of these equalizers under different spectral efficiency settings is evaluated in more detail in the following.

IV. Numerical Results

In this section, we numerically evaluate the performance of the proposed approach. In the following experiments, the results are averaged over a large number of simulations and the chirp duration is fixed at $\tau_d = 1\mu s$. Other parameters are specified for each setting.

First we study the error behavior for different intervals using our BER expression as well as empirical results. The bandwidth is set to 100 MHz and the symbol interval τ varies proportionally to the Nyquist interval $\tau_N = \frac{1}{B} = 0.01\mu s$. As can be seen from the Fig. 2, when the interval is large,

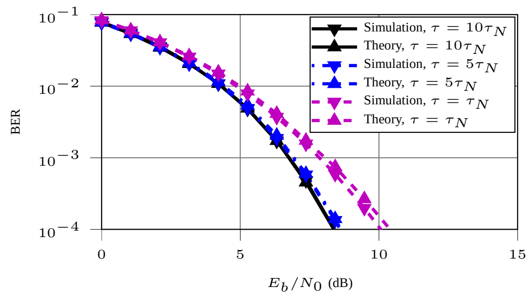

Fig. 2: Bit error rate versus signal to noise per bit with varying symbol interval τ. The bandwidth $B = 100$ MHz.

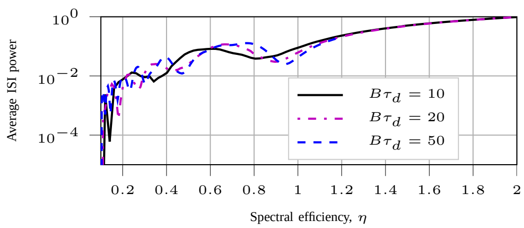

Fig. 3: Average ISI power with varying time-bandwidth product $\tau_d B$.

then the resulting BER is close to the theoretical BER for BPSK in an AWGN channel, due to small ISI. In contrast, the BER degrades dramatically when the ISI is more severe, for instance, $\tau = \tau_N$. Also, the analytical BER is sufficiently accurate under low and moderate ISI, but not at large ISI, due to the approximation error of the Q-function.

In the second experiment, we study the possibility of achieving the rate close to the Nyquist rate without utilizing any equalization methods. More specifically, we take advantages of the formulation (8) to visualize the average ISI power when the spectral efficiency η increases. Generally speaking, the ISI is severe as shown in Fig. 3 when the data rate is close to or more than the Nyquist rate. Interestingly, Fig. 3 also shows that we can always find a local minimum close to the Nyquist rate, but with lower average ISI power. For instance, for 10 MHz, the average ISI at the point $\eta = 0.8$, or equivalently $\tau = 1.25\tau_N$ has much lower ISI than for $\tau = \tau_N$. As a result, we may use this minimum without additional equalizers, as investigated in the following experiment.

Now we turn our attention to the effects of local minimum of the ISI mentioned above, with $B = 10$ MHz. In Fig. 3 we see that $\tau = 1.25\tau_N$ reduces ISI significantly in comparison with $\tau = \tau_N$. As a result, its BER outperforms that of τ_N as shown in Fig. 4. However, a gap between this local minimum and the no-ISI curve still remains, and, therefore, efficient equalization is vital to compensate for this gap, as demonstrated in the following simulation.

Next, we investigate the effectiveness of equalization under Nyquist rate. We set the bandwidth to 10 MHz and 50 MHz and the symbol interval to τ_N. Unsurprisingly, the equalization has greatly improved the performance of the system, although the ISI is severe under this setting. More specifically, the BER approaches that of the BPSK without ISI as expected. Note that the performance of ZF and MMSE are quite similar due to the fact that the channel matrix is positive definite for the considered τ.

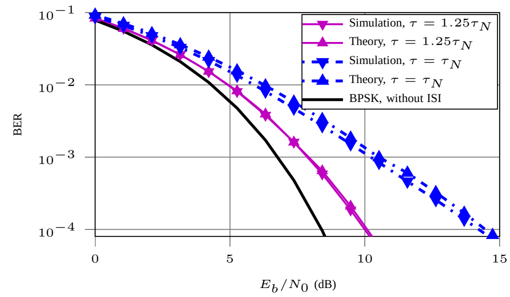

Fig. 4: Bit error rate versus signal to noise per bit in case of close to Nyquist rate and without equalization. The bandwidth $B = 10$ MHz.

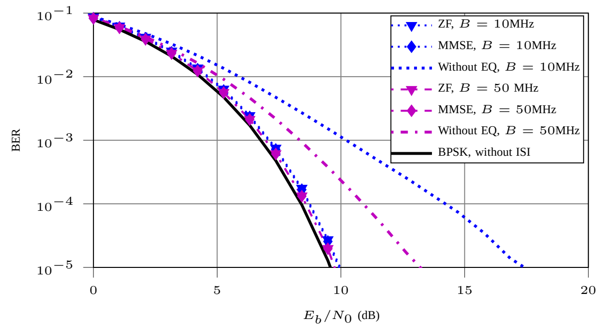

Fig. 5: Bit error rate versus signal in cases of Nyquist rate.

In principle, the design of a linear equalizer utilizing known chirp autocorrelation function can be applicable to any rate. With this in mind, in the last simulation, we demonstrate that a faster-than-Nyquist rate using chirps is also achievable. For the considered simulation, the number of chirps is transmitted at 20% and 50% above the Nyquist rate. In Fig. 6, the performance without equalization suffers from severe ISI. As mentioned in the preceding section, linear equalizers may reduce the ISI significantly but their BERs are not very close to that of Nyquist rate (c.f. Fig. 5). Interestingly, this simulation shows that the equalization for this setting is more effective when the mainlobe width is large, e.g., 10 MHz. We also note that more research efforts should be put into investigating the maximum achievable rate and designing more efficient non-linear equalizers for this important setting.

V. Conclusions and Future Works

We have derived a simple BER expression to evaluate the performance of an overlapped chirp-based system using the

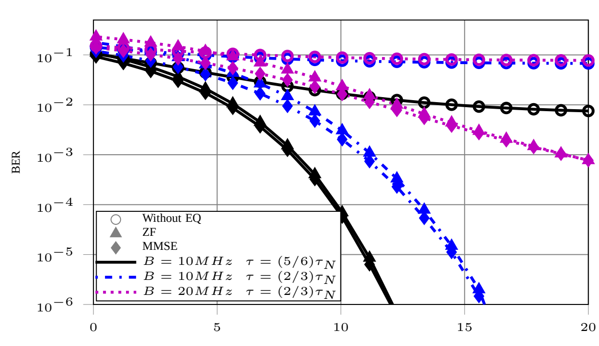

Fig. 6: Bit error rate for a 20% and 50% faster rate than Nyquist rate.

series of exponentially decreasing cosine. More importantly, we have reformulated the receiving signals to arrive at simple equalization techniques which can even work efficiently at a faster rate than Nyquist rate. In fact, our BER expression has provided sufficient accuracy compared to the existing formulation. Moreover, our extensive simulation and analysis have demonstrated the effectiveness of our equalization approach, which have reduced ISI components significantly and the bit error rate thus approaches the one without ISI.

Having shown the potential of utilizing chirps for high-rate communications, the research can be explored in broader scenarios taking into account, for instance, the effect of a multipath channel model. More importantly, the guard interval can be replaced by a sequence of non-overlapping chirps, used for not only radar detection but also channel estimation and/or synchronization, which in turn allows easy cooperation and flexible resource allocation. Future research on this important joint radar and communications systems will address key issues such as interference management, parameter identifiability as well as resource allocation strategy [7], [14].

Appendix A
Proof of Zero-ISI Condition for Chirp Overlapping

For zero ISI, it is required that $r(k\tau) = 0, \forall k > 0$. As a result, τ must be a root of the autocorrelation function (5), and the n-th root is given by

$$\tau_0(n) = \frac{\tau_d}{2}\left(1 - \sqrt{1 - \frac{4n}{B\tau_d}}\right), \quad \text{with } B\tau_d > 4n. \tag{29}$$

Note that $n > 0$ and $k > 0$ without loss of generality. Considering (5) and the zero-ISI condition, then to obtain the shortest interval τ with zero ISI, we need to find the smallest integer n such that

$$\frac{Bk\tau_0(n)\left(\tau_d - |k|\tau_0(n)\right)}{\tau_d} \in \mathbb{Z}, \quad \forall k \in \mathbb{Z}. \tag{30}$$

Now, for $k > 0$, substituting (29) into (30), we get the following condition

$$k^2 n + \frac{k(k-1)}{2}\left(\sqrt{B^2\tau_d^2 - 4nB\tau_d} - B\tau_d\right) \in \mathbb{Z}, \tag{31}$$

which, since $k, n \in \mathbb{Z}$, this simplifies to

$$\sqrt{B^2\tau_d^2 - 4nB\tau_d} - B\tau_d \in \mathbb{Z}. \tag{32}$$

We can easily see that the proposed interval in [11] i.e., $\tau = \sqrt{\tau_d/B}$ can be achieved only and only if $n = \sqrt{B\tau_d} - 1$ and thus $\sqrt{B\tau_d} \in \mathbb{Z}$. This result immediately satisfies the condition of (32). However, this condition is impractical since time-bandwidth product can be arbitrary number and thus non-orthogonality is unavoidable.

Appendix B
BER Calculation Proof

Recall that the bit error rate of the system is defined as

$$P_e = \mathbb{E}(P_{e|v_l}). \tag{33}$$

Therefore, we can take the bit error rate as a result of averaging all possibilities. However, this approach is not feasible, if impossible, if the number of possibilities is too large. Instead, we may employ an approximation of the Q-function so that the computation of the expectation is practicable. In this paper, we rely on the series of exponentially decreasing cosine [5], [12] to approximate BER expression, which is given by

$$Q(x) \approx \Re\left[\sum_{j=0}^{n_T-1} c_j e^{(\lambda_j + i\omega_j)x}\right]. \tag{34}$$

Substituting (34) into (33), we obtain the following

$$\begin{aligned} P_e &= \Re\left[\sum_{j=0}^{n_T-1} c_j \mathbb{E}\left(e^{(\lambda_j+i\omega_j)\sqrt{\frac{2E_b}{N_0}}(1-v_l)}\right)\right] && (35)\\ &= \Re\left[\sum_{j=0}^{n_T-1} c_j e^{(\lambda_j+i\omega_j)\sqrt{\frac{2E_b}{N_0}}}\mathbb{E}\left(\mathbf{\Omega}\right)\right] && (36) \end{aligned}$$

where $\mathbf{\Omega} = e^{-(\lambda_j+i\omega_j)\sqrt{\frac{2E_b}{N_0}}\sum_{k=0,k\neq l}^{(\kappa-1)} v_{-(k-l)}}$.

As $\xi = \pm 1$, we thus get

$$P_e = \Re\left[\sum_{j=0}^{n_T-1} c_j e^{\rho}\mathbb{E}\left(\prod_{k=0,k\neq l}^{(\kappa-1)} \cosh(r((l-k)\tau)\rho)\right)\right] \tag{37}$$

where $\rho = (\lambda_j + i\omega_j)\sqrt{\frac{2E_b}{N_0}}$, which completes the proof.

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